ORBIT BRAID ACTION ON A FINITE GENERATED GROUP

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Abstract. This paper aims to generalize Artin's ideas in [Art25] to establish an one-to-one correspondence between the orbit braid group $B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)$ and a quotient of a group formed by some particular $\mathbb{Z}_p$-homeomorphisms of a punctured plane. First, we find a faithful representation of $B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)$ in a finite generated group whose generators are corresponding to generators of fundamental group of the punctured plane, by demonstrating that the representation from $B_n^{orb}(\mathbb{C}^\times, \mathbb{Z}_p)$ to the fundamental group is faithful. Next we investigate some characterizations of orbit braid representation to get our conclusion.

1. Introduction

Braid groups are fundamental objects in mathematics, which were first defined rigorously and studied by Artin in [Art25]. The most typical braid group is $B_n = \pi_1(F(\mathbb{C}, n)/\Sigma_n)$, where

$$F(\mathbb{C}, n) = \{(x_1, \ldots, x_n) \in \mathbb{C}^{\times n} | x_i \neq x_j \text{ for } i \neq j\}$$

and Σ_n is the free action of the symmetric group on $F(\mathbb{C}, n)$, defined by $\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. An element of B_n is an isotopy equivalent class of realized braids in 3−dimension space. By identifying the initial points and the end points of these braids, we could get every knot and link in S^3, according to Alexander Theorem [Ale23].

Artin also found an algorithm to compute link group, which is the fundamental group of link's complement in S^3, and has been regarded as one of the most important invariants of links in S^3. Artin's method is as followed (the third step was simplified by Birman in [BC74]):

- Established a faithful representation of B_n in the fundamental group of $\mathbf{D}^2 \setminus Q_n$(the standard disk in $\mathbb{C}$ punctured at n distinct points).
- Investigate some characteristics of homeomorphisms of $\mathbf{D}^2$ that fix boundary, to identify each such kind of homeomorphism with a braid element.
- Using $\beta : \mathbf{D}^2 \setminus Q_n \to \mathbf{D}^2 \setminus Q_n$ to represent $\beta \in B_n$ through the identification, we put the quotient space $\{(\mathbf{D}^2 \setminus Q_n) \times I\}/\sim$ with $(z, 0) \sim (\beta(z), 1)$ into S^3. Use the fibration $(\mathbf{D}^2 \setminus Q_n) \times I \to S^1$ to find out that the action of β on the punctured disk is exactly the presentation of link group of the closure of $\bar{\beta}$.

The theory of orbit braids was upbuilt by Hao Li, Zhi Lü and Fenglin Li in [LLL19]:

Definition 1.1. for a connected topological manifold M of dimension greater than one, which admits an effective G-action where G is a finite group,

$$F_G(M, n) = \{(x_1, \ldots, x_n) \in M^{\times n} | G(x_i) \cap G(x_j) = \emptyset \text{ for } i \neq j\}.$$

Then $\alpha : I \to F_G(M, n)$ with $\alpha(0) = \mathbf{x} = (x_1, \ldots, x_n)$ and $\alpha(1) = g\mathbf{x}_\sigma = (g_1 x_{\sigma(1)}, \ldots, g_n x_{\sigma(n)})$ for some $(g, \sigma) \in G^{\times n} \times \Sigma_n$ is called an orbit braid in $M \times I$.

They have defined an equivalence relation among all orbit braids at an orbit base point, so that all equivalence classes can form a group $\mathcal{B}_n^{orb}(M, G)$:

Definition 1.2. We say that two paths $\alpha, \beta : I \to F_G(M, n)$ with the same initial points are isotopic with respect to the G-action relative to endpoints in $M \times I$ if $\alpha \simeq \beta$ rel ∂I.

Intuitively, a string of an orbit braid could pass through its own orbits but could not pass through other strings and their orbits. The geometric presentation of classical braid group $B_n(\mathbb{R}^2)$ in $\mathbb{R}^2 \times I$ gives us much more insights to the case of orbit braid group. Thus we begin our work from the case of $\mathbb{C} \approx \mathbb{R}^2$ and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ with the typical action $\mathbb{Z} \curvearrowright \mathbb{C}$ defined by $(n, z) \mapsto \mathrm{e}^{\frac{2n\pi i}{p}} z$.

According to [LLL19], the generators of $B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)$ are b and $b_k(k = 0, \ldots, n-2)$ where b_k could be realized as a representative path

$$\alpha_k(t) = (1 + \mathrm{i}, \ldots, (k+1) + (k+2)\mathrm{i} + \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}(1-t)}, (k+2) + (k+1)\mathrm{i} + \mathrm{e}^{\frac{\pi}{2}\mathrm{i}(1+t)}, \ldots, n + n\mathrm{i}),$$

and b could be realized as a representative path

$$\alpha(t) = ((1 + \mathrm{i})\mathrm{e}^{\frac{t}{p}}, 2 + 2\mathrm{i}, \ldots, n + n\mathrm{i}).$$

And $B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)$ admits several relations:

(a) $b^p = e$;
(b) $(bb_1)^p = (b_1 b)^p$ (p even);
(c) $b_k b = b b_k (k > 1)$;
(d) $b_k b_{k+1} b_k = b_{k+1} b_k b_{k+1}$;
(e) $b_k b_l = b_l b_k$ ($|k - l| > 1$).

$B_n^{orb}(\mathbb{C}^\times, \mathbb{Z}_p)$ has just the same gererators but one less relation. b^p is a non-trivial element in this group, since the first string cannot pass through the central point of the plane.

Then, we tried to employ Artins method, which computes the fundamental group of the complement of the closure of the ordinary braid, to the case of orbit braid. We have made generalizations of the first two steps of Artin:

- We established a faithful representation of $B_n^{orb}(\mathbb{C}^\times, \mathbb{Z}_p)$ in the fundamental group of $\mathbf{D}^2 \setminus Q_p n$, and by which we proved a representation of $B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)$ in a finite generated group is faithful.
- Then we investigated some characteristics of G-homeomorphisms of $\mathbf{D}^2$ that fix boundary, to identify each such kind of G-homeomorphism with a braid element of $B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)$.

2. Orbit braid action on a finite generated group

Definition 2.1. R_{pn} is a group with generators x_{ij}, $0 \le i \le p-1$, $0 \le j \le n-1$, and presentation given by

$$\langle x_{00}, x_{01}, \cdots x_{p-1,n-1} | (\prod_{k=i}^{i+p-1} x_{k \bmod p, 0}) x_{ij} (\prod_{k=i}^{i+p-1} x_{k \bmod p, 0})^{-1} = x_{ij}, 0 \le i \le p-1, 0 < j \le n-1 \rangle$$

ρ_R is a homomorphism from $B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)$ to $Aut R_{pn}$, such that

$$\rho_R(b) : \begin{cases} x_{i0} \mapsto x_{(i+1) \bmod p, 0}, & 0 \le i \le p-1 \\ x_{ij} \mapsto x_{i0}^{-1} x_{ij} x_{i0}, & 0 \le i \le p-1, j \ne 0 \end{cases}$$

$$\rho_R(b_k) : \begin{cases} x_{ik} \mapsto x_{i,k+1}, & 0 \le i \le p-1 \\ x_{i,k+1} \mapsto x_{i,k+1} x_{ik} x_{i,k+1}^{-1}, & 0 \le i \le p-1 \\ x_{ij} \mapsto x_{ij}, & 0 \le i \le p-1, j \ne k, k+1 \end{cases}$$

Theorem 2.2. *Let*

$$M_n^{orb} = \{Aut(\pi_1(\mathbf{D}^2 \setminus Q_{pn})) \cap Hom_G(\mathbf{D}^2 \setminus Q_{pn}, Id_{\partial \mathbf{D}^2})\} / < \rho_R(b^p) >,$$

where Q_{pn} is the set of n points located at $(1,1), (2,2) \cdots (n,n)$ and their orbits under $\mathbb{Z}_p$ in the interior of 2-ball $\mathbf{D}^2$, and G-homeomorphisems are identity on the $\partial \mathbf{D}^2$, and $< \rho_R(b^p) >$ is generated by a G-homeomorphisem which fixes outside $B_{\frac{5}{3}}(0,0)$ and is 2π rotation within $B_{\frac{4}{3}}(0,0)$. Then

$$M_n^{orb} \cong \rho_R(B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)) \cong B_n^{orb}(\mathbb{C}, \mathbb{Z}_p).$$

In fact, x_{ij} in Definition 2.1 is a basis of $\pi_1(\mathbf{D}^2 \setminus Q_{pn})$ generated by a loop enclosing $q_{ij} \in Q_{pn}$, with the base point $(0,0)$.

Lemma 2.3. ρ_R *is well defined.*

Proof. Verify $\rho_R(b^p)=\rho_R(e)$, whenever $0\le i,j\le p-1, j\neq 0$:

$$\rho_R(b^p):\begin{cases} x_{i0}\overset{\rho_R(b)}{\mapsto}x_{(i+1)\bmod p,0}\overset{\rho_R(b)}{\mapsto}\cdots\overset{\rho_R(b)}{\mapsto}x_{(i+p)\bmod p,0}=x_{i0}\\ x_{ij}\overset{\rho_R(b)}{\mapsto}x_{i0}^{-1}x_{ij}x_{i0}\overset{\rho_R(b)}{\mapsto}x_{(i+1)\bmod p,0}^{-1}(x_{i0}^{-1}x_{ij}x_{i0})x_{(i+1)\bmod p,0}\overset{\rho_R(b)}{\mapsto}\cdots\\ \overset{\rho_R(b)}{\mapsto}x_{(i+p-1)\bmod p,0}^{-1}\cdots(x_{(i+p)\bmod p,0}^{-1}x_{(i+p)\bmod p,j}x_{(i+p)\bmod p,0})\cdots x_{(i+p-1)\bmod p,0}\\ =(\prod_{k=i}^{i+p-1}x_{k\bmod p,0})^{-1}x_{ij}(\prod_{k=i}^{i+p-1}x_{k\bmod p,0})^{-(-1)}=x_{ij}\end{cases}$$

Verify $\rho_R((bb_0)^p)=\rho_R((b_0b)^p)$, whenever $0\le i,j\le p-1, j\neq 0,1$, p even :

$$\rho_R((bb_0)^p):\begin{cases} x_{i0}\overset{\rho_R(b)}{\mapsto}x_{(i+1)\bmod p,0}\overset{\rho_R(b_0)}{\mapsto}x_{(i+1)\bmod p,1}\overset{\rho_R(b)}{\mapsto}x_{(i+1)\bmod p,0}^{-1}x_{(i+1)\bmod p,1}x_{(i+1)\bmod p,0}\\ \overset{\rho_R(b_0)}{\mapsto}x_{(i+1)\bmod p,1}^{-1}(x_{(i+1)\bmod p,1}x_{(i+1)\bmod p,0}x_{(i+1)\bmod p,1}^{-1})x_{(i+1)\bmod p,1}\\ =x_{(i+1)\bmod p,0}\overset{\rho_R(b)}{\mapsto}\cdots\overset{\rho_R(b_0)}{\mapsto}\cdots\cdots\overset{\rho_R(b)}{\mapsto}\cdots\overset{\rho_R(b_0)}{\mapsto}x_{(i+\frac{p}{2})\bmod p,0};\\ x_{i1}\overset{\rho_R(b)}{\mapsto}x_{i0}^{-1}x_{i1}x_{i0}\overset{\rho_R(b_0)}{\mapsto}x_{i0}\overset{\rho_R(b)}{\mapsto}\cdots\cdots\overset{\rho_R(b_0)}{\mapsto}x_{(i+\frac{p}{2})\bmod p,1};\\ x_{ij}\overset{\rho_R(b)}{\mapsto}x_{i0}^{-1}x_{ij}x_{i0}\overset{\rho_R(b_0)}{\mapsto}x_{i1}^{-1}x_{ij}x_{i1}\overset{\rho_R(b)}{\mapsto}(x_{i0}^{-1}x_{i1}x_{i0})^{-1}(x_{i0}^{-1}x_{ij}x_{i0})(x_{i0}^{-1}x_{i1}x_{i0})\\ =x_{i0}^{-1}(x_{i1}^{-1}x_{ij}x_{i1})x_{i0}\overset{\rho_R(b_0)}{\mapsto}x_{i1}^{-1}(x_{i1}x_{i0}x_{i1}^{-1})^{-1}x_{ij}(x_{i1}x_{i0}x_{i1}^{-1})x_{i1}=x_{i0}^{-1}x_{i1}^{-1}x_{ij}x_{i1}x_{i0}\\ \overset{\rho_R(b)}{\mapsto}\cdots\cdots\overset{\rho_R(b_0)}{\mapsto}(\prod_{k=i}^{i+\frac{p}{2}-1}(x_{k(\bmod p),1}x_{k(\bmod p),0}))^{-1}x_{ij}(\prod_{k=i}^{i+\frac{p}{2}-1}(x_{k(\bmod p),1}x_{k(\bmod p),0}));\end{cases}$$

$$\rho_R((b_0b)^p):\begin{cases} x_{i0}\overset{\rho_R(b_0)}{\mapsto}x_{i1}\overset{\rho_R(b)}{\mapsto}x_{i0}^{-1}x_{i,1}x_{i0}\overset{\rho_R(b_0)}{\mapsto}x_{i0}\overset{\rho_R(b)}{\mapsto}x_{(i+1)\bmod p,0}\\ \overset{\rho_R(b_0)}{\mapsto}\cdots\overset{\rho_R(b)}{\mapsto}\cdots\cdots\overset{\rho_R(b_0)}{\mapsto}\cdots\overset{\rho_R(b)}{\mapsto}x_{(i+\frac{p}{2})\bmod p,0};\\ x_{i1}\overset{\rho_R(b_0)}{\mapsto}x_{i1}x_{i0}x_{i1}^{-1}\overset{\rho_R(b)}{\mapsto}(x_{i0}^{-1}x_{i1}x_{i0})x_{(i+1)\bmod p,0}(x_{i0}^{-1}x_{i1}x_{i0})^{-1}\overset{\rho_R(b_0)}{\mapsto}x_{i0}x_{(i+1)\bmod p,1}x_{i0}^{-1}\\ \overset{\rho_R(b)}{\mapsto}x_{(i+1)\bmod p,1}\overset{\rho_R(b_0)}{\mapsto}\cdots\cdots\overset{\rho_R(b)}{\mapsto}x_{(i+\frac{p}{2})\bmod p,1};\\ x_{ij}\overset{\rho_R(b_0)}{\mapsto}x_{ij}\overset{\rho_R(b)}{\mapsto}x_{i0}^{-1}x_{ij}x_{i0}\overset{\rho_R(b_0)}{\mapsto}x_{i1}^{-1}x_{ij}x_{i1}\overset{\rho_R(b)}{\mapsto}x_{i0}^{-1}x_{i1}^{-1}x_{ij}x_{i1}x_{i0}\\ \overset{\rho_R(b_0)}{\mapsto}\cdots\cdots\overset{\rho_R(b)}{\mapsto}(\prod_{k=i}^{i+\frac{p}{2}-1}(x_{k(\bmod p),1}x_{k(\bmod p),0}))^{-1}x_{ij}(\prod_{k=i}^{i+\frac{p}{2}-1}(x_{k(\bmod p),1}x_{k(\bmod p),0}));\end{cases}$$

$\rho_R(b_kb)=\rho_R(bb_k)$ $(k>0)$, $\rho_R(b_kb_{k+1}b_k)=\rho_R(b_{k+1}b_kb_{k+1})$, $\rho_R(b_kb_l)=\rho_R(b_lb_k)$ $(|k-l|>1)$ are easy to check. ☐

We first consider $B_n^{orb}(\mathbb{C}^\times,\mathbb{Z}_p)$ which has the same generators as $B_n^{orb}(\mathbb{C},\mathbb{Z}_p)$. Let F_{pn} be a free group with the same generators of R_{pn}, and $\rho_F: B_n^{orb}(\mathbb{C}^\times,\mathbb{Z}_p)\to F_{pn}$ just as what ρ_R does. Our aim is to prove that ρ_F is faithful first and then ρ_R.

But before this, we have to make some preparations. Let $F^m_{\mathbb{Z}_p}\mathbb{C}^\times\subset\mathbb{C}^\times$ be a set of fixed distinguished m points and their orbits. Define $F^m_{\mathbb{Z}_p}(\mathbb{C}^\times,n)=F_{\mathbb{Z}_p}(\mathbb{C}^\times\setminus F^m_{\mathbb{Z}_p}\mathbb{C}^\times,n)$. Let $\pi^r_n: F^m_{\mathbb{Z}_p}(\mathbb{C}^\times,n)\to F^m_{\mathbb{Z}_p}(\mathbb{C}^\times,r)$ be the projection introduced from $(\mathbb{C}^\times)^n=(\prod_{i=r+1}^{n}\mathbb{C}^\times)\times(\prod_{i=1}^{r}\mathbb{C}^\times)=(\mathbb{C}^\times)^{n-r}\times(\mathbb{C}^\times)^r\to(\mathbb{C}^\times)^r$. The projection π^{n-1}_n is a fibration with fiber $F^{n-1}_{\mathbb{Z}_p}\mathbb{C}^\times$, whose proof is similar to [FN62]. Then we have

Lemma 2.4. $\pi_2(F_{\mathbb{Z}_p}(\mathbb{C}^\times,n-1))=1$.

Proof. By the covering homotopy property of fibration we have an exact sequence

$$\pi_2(F^{n-1}_{\mathbb{Z}_p}\mathbb{C}^\times)\longrightarrow\pi_2(F_{\mathbb{Z}_p}(\mathbb{C}^\times,n))\xrightarrow{\pi_1(\pi_n^{n-1})}\pi_2(F_{\mathbb{Z}_p}(\mathbb{C}^\times,n-1))\ .$$

$\pi_2(F^{n-1}_{\mathbb{Z}_p}\mathbb{C}^\times)$ is definitely trivial since $F^{n-1}_{\mathbb{Z}_p}\mathbb{C}^\times$ has the same homotopy type as wedge of $p(n-1)+1$ circles, whose universal covering space is a infinite tree in $(p(n-1)+1)$-dimension space. The conclusion follows the induction of n. ☐

Lemma 2.5. ρ_F *is faithful.*

Proof. [LLL19] has proved that

$$1 \to P_n(\mathbb{C}^\times, \mathbb{Z}_p) \to B_n^{orb}(\mathbb{C}^\times, \mathbb{Z}_p) \to \mathbb{Z}_p^{\times n} \rtimes \Sigma_n \to 1$$

is a short exact sequence. Define $Aut_{\mathbb{Z}_p} F_n \subset Aut F_n$ such that every $h \in Aut_{\mathbb{Z}_p} F_n$ satisfies

$$c(h(x_{ij})) = h(x_{i+1(\mod p),j})$$

where $c \in Aut F_{pn}$ subjects to $c(x_{ij}) = x_{i+1(\mod p),j}$. Now we have two short exact rows and they form a commutative diagram

$$\begin{array}{ccccc} P_n(\mathbb{C}^\times, \mathbb{Z}_p) & \longrightarrow & B_n^{orb}(\mathbb{C}^\times, \mathbb{Z}_p) & \longrightarrow & \mathbb{Z}_p^{\times n} \rtimes \Sigma_n \\ \downarrow{\scriptstyle \rho_F|_{P_n(\mathbb{C}^\times, \mathbb{Z}_p)}} & & \downarrow{\scriptstyle \rho_F} & & \downarrow{\scriptstyle \cong} \\ \ker\rho & \longrightarrow & Aut_{\mathbb{Z}_p} F_n & \longrightarrow & Aut_{\mathbb{Z}_p}(F_n/[F_n, F_n]) \end{array}$$

By Five Lemma we just need to check that $\rho_F|_{P_n(\mathbb{C}^\times, \mathbb{Z}_p)}$ is monomorphism.

With the idea that the central point should be regarded as a fixed vertical string, We could easily see that the group $P_n(\mathbb{C}^\times, \mathbb{Z}_p)$ has generators (see Figure 1)

$$\begin{gathered} \{A_{iqj} = \{(b_{i-1} \cdots b_0)b(b_0 \cdots b_{i-1})\}^q b_i \cdots b_{j-1} b_j^2 b_{j-1}^{-1} \cdots b_i^{-1} \{(b_{i-1} \cdots b_0)b(b_0 \cdots b_{i-1})\}^{-q} : \\ 0 \le i < j \le n-1, 0 \le q \le p-1, \\ A_i = (b_{i-1} \cdots b_0 b b_0 \cdots b_{i-1})^p : 0 \le i \le n-1\} \end{gathered}$$

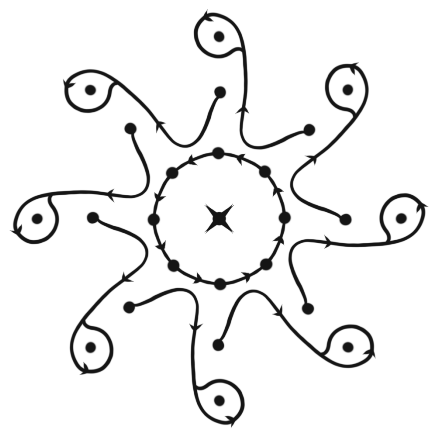

Figure 1. $A_{0,1,1} = b_0 b b_0 b_1^2 (b_0 b b_0)^{-1}$ and $A_0 = b^p$

Note that $P_{n-1}(\mathbb{C}^\times, \mathbb{Z}_p) = \{A_{iqj} : 0 < i < j \le n-1, 0 \le q \le p-1; A_i : i > 0\}$ is a subgroup of $P_n(\mathbb{C}^\times, \mathbb{Z}_p)$. Define a projection

$$\begin{aligned} \epsilon : P_n(\mathbb{C}^\times, \mathbb{Z}_p) &\to P_{n-1}(\mathbb{C}^\times, \mathbb{Z}_p) \\ A_{iqj} &\mapsto \begin{cases} A_{iqj} & if\ 0 < i < j \le n-1 \\ 1 & if\ 1 \le j \le n-1, i = 0 \end{cases} \\ A_i &\mapsto \begin{cases} A_i & if\ 0 < i \le n-1 \\ 1 & if\ i = 0 \end{cases} \end{aligned}$$

According to [LLL19], there is an isomorphism $l_n : P_n(\mathbb{C}^\times, \mathbb{Z}_p) \to \pi_1(F_{\mathbb{Z}_p}(\mathbb{C}^\times, n), \mathbf{x})$.

Now we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_n(\mathbb{C}^\times, \mathbb{Z}_p) = ker\epsilon & \longrightarrow & P_n(\mathbb{C}^\times, \mathbb{Z}_p) & \xrightarrow{\epsilon} & P_{n-1}(\mathbb{C}^\times, \mathbb{Z}_p) & \longrightarrow & 1 \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle l_n|_{ker\epsilon}} & & \downarrow{\scriptstyle l_n} & & \downarrow{\scriptstyle l_{n-1}} & & \downarrow{\scriptstyle id} \\ 1 & \longrightarrow & \pi_1(F^{n-1}_{\mathbb{Z}_p}\mathbb{C}^\times) & \longrightarrow & \pi_1(F_{\mathbb{Z}_p}(\mathbb{C}^\times, n)) & \xrightarrow{\pi_1(\pi_n^{n-1})} & \pi_1(F_{\mathbb{Z}_p}(\mathbb{C}^\times, n-1)) & \longrightarrow & 1 \end{array}$$

where the rows are exact, and $\pi_1(F^{n-1}_{\mathbb{Z}_p}\mathbb{C}^\times) = \pi_1(\mathbb{C} \setminus Q_{p(n-1)+1})$, and the bottom left corner is proved by Lemma 2.4. By Five Lemma, $\ker \epsilon$ is isomorphic to $\pi_1(F^{n-1}_{\mathbb{Z}_p}\mathbb{C}^\times)$. This means that $U_n(\mathbb{C}^\times, \mathbb{Z}_p)$ is a free group with a free basis $\{A_0, A_{0qj} : 0 < j \leq n-1, 0 \leq q \leq p-1\}$ since $\{l_n(A_0, l_n(A_{0qj} : 0 < j \leq n-1, 0 \leq q \leq p-1\}$ is a free basis of $\pi_1(F^{n-1}_{\mathbb{Z}_p}\mathbb{C}^\times)$.

Intuitively, we could examine what $A_{i(qj)}A_{0rk}A^{-1}_{i(qj)}$ is, to see why $\ker \epsilon$ is generated by $\{A_0, A_{0qj} : 0 < j \leq n-1, 0 \leq q \leq p-1\}$. Just shift the loop of A_{0rk} down through the loop of $A_{i(qj)}$, during which the initial point and end point of the first loop is maintained so that it can always been presented as composition of A_{0ab} and A_0. Than we just reduce the loops of $A_{i(qj)}$ and $A^{-1}_{i(qj)}$ to identity. For example, $A_1A_{001} = (A_0A_{001}A_0^{-1})A_1$. Other cases could be sorted and handled by induction. As a result, every element $a \in P_n(\mathbb{C}^\times, \mathbb{Z}_p)$ can be represented uniquely as the normal form

$$a = a_2 \cdots a_n$$

with $a_j \in U_j(\mathbb{C}^\times, \mathbb{Z}_p)$.

We have seen that $P_n(\mathbb{C}^\times, \mathbb{Z}_p)$ acts by conjugation as a group of automorphisms of $U_{n+1}(\mathbb{C}^\times, \mathbb{Z}_p)$. The automorphisms of fundamental groups could be regarded as the transformations of the loops of $U_{n+1}(\mathbb{C}^\times, \mathbb{Z}_p)$, as the same fashion discussed above. Thus if we define $f : U_{n+1}(\mathbb{C}^\times, \mathbb{Z}_p) \to F_n$ by $f(A_{0qj}) = x_{qj}(0 \leq q \leq p-1, 0 < j \leq n-1), f(A_0) = 0$, the conjugate action would make the following diagram commutative:

$$\begin{array}{ccc} P_n(\mathbb{C}^\times, \mathbb{Z}_p) & \xrightarrow{c} & AutU_{n+1}(\mathbb{C}^\times, \mathbb{Z}_p) \ . \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle Aut(f)} \\ P_n(\mathbb{C}^\times, \mathbb{Z}_p) & \xrightarrow{\rho_F|_{P_n(\mathbb{C}^\times, \mathbb{Z}_p)}} & Aut_{\mathbb{Z}_p}F_n \end{array}$$

Because the image of c leaves A_0 constant, we conclude that $\ker \rho_F|_{P_n(\mathbb{C}^\times, \mathbb{Z}_p)}$ is the subgroup of elements in $P_n(\mathbb{C}^\times, \mathbb{Z}_p)$ which commute with $U_{n+1}(\mathbb{C}^\times, \mathbb{Z}_p)$.

Assume that there is an element $a \in \ker \rho_F|_{P_n(\mathbb{C}^\times, \mathbb{Z}_p)}$ such that $a = a_2 \cdots a_i$. Then a_i commute with A_{00i}. Let $\pi = b_0b_1 \cdots b_{i-1}$. We have

$$\begin{aligned} \pi A_{00i}\pi^{-1} &= A_{001} \\ a_iA_{00i}a_i^{-1} &= A_{00i} \\ (\pi a_i\pi^{-1})A_{001}(\pi a_i\pi^{-1})^{-1} &= (\pi a_i\pi^{-1})(\pi A_{00i}\pi^{-1})(\pi a_i\pi^{-1})^{-1} \\ &= \pi a_iA_{00i}a_i^{-1}\pi^{-1} \\ &= \pi A_{00i}\pi^{-1} \\ &= A_{001} \end{aligned}$$

However, $\pi a_i\pi^{-1}$ is in $U_{n+1}(\mathbb{C}^\times, \mathbb{Z}_p)$. It must be A^l_{001} since it commutes with A_{001}. Then $a_i = \pi^{-1}A^l_{001}\pi = A^l_{00i}$, which is a contradiction. □

Now we consider ρ_R.

Theorem 2.6. *ρ_R is faithful.*

Proof. If $\beta \in B^{orb}_n(\mathbb{C}, \mathbb{Z}_p)$ and $\rho_R(\beta) = id$, and β' is the corresponding element in $B^{orb}_n(\mathbb{C}^\times, \mathbb{Z}_p)$, then $\rho_F(\beta')(x_{ij}) = A_{ij}x_{\mu_{ij}}A^{-1}_{ij}$ must be x_{ij} in R_{pn}. Because when the relation of R_{pn} is reduced, the odevity of words is kept, μ_{ij} must be ij. If $j > 0$, A_{ij} must be multiple of $(\prod_{k=i}^{i+p-1} x_{k \bmod p,0})$ since other words don't commute with x_{ij}. If $j = 0$, since the words on the right and the left of x_{ij} are symmetric, x_{ij} cannot commute with other words as a part of $(\prod_{k=i}^{i+p-1} x_{k \bmod p,0})$. Thus A_{i0} must be 1.

Note that

$$\rho_F(b)(\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij})) = (x_{00})^{-1}\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij})x_{00}$$

$$\rho_F(b_k)(\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij})) = \prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij}).$$

Thus

$$\rho_F(\beta')(\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij})) = A\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij})A^{-1}.$$

Thus A_{ij} would contain the same number of $(\prod_{k=i}^{i+p-1} x_{k \bmod p,0})$. Suppose $A_{ij} = (\prod_{k=i}^{i+p-1} x_{k \bmod p,0})^m$. In conclusion, $\rho_F(\beta')$ would act just as $\rho_F((b^p)^m)$. Because we have proved ρ_F is faithful, β' thus β must be $(b^p)^m$. So β is identity in $B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)$. □

3. Characterizations of orbit braid representation

Lemma 3.1. *If $\rho \in AutR_{pn} \setminus \{id_{R_{pn}}\}$: (1)could be described as a simplest form:*

$$\rho(x_{ij}) = A_{ij}x_{\mu_{ij}}A_{ij}^{-1} \tag{3.i}$$

where $(i,j) \mapsto \mu_{ij}$ is a permutation, and (2)

$$\rho(\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij})) = A\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij})A^{-1} \tag{3.ii}$$

and (3)

$$c(A_{ij}) = A_{i+1(\bmod p),j} \tag{3.iii}$$

$$c(x_{\mu_{ij}}) = x_{\mu_{i+1(\bmod p),j}} \tag{3.iv}$$

where $c \in AutR_{pn}$ subjects to $c(x_{ij}) = x_{i+1(\bmod p),j}$. Then either (a) there is a pair (i,j) such that $x_{\mu_{ij}}A_{ij}^{-1}$ could be absorbed by $A_{f(ij)}$; or (b) there is a pair (i,j) such that A_{ij}^{-1} could absorb $A_{f(ij)}x_{\mu_{f(ij)}}$, where $f(ij)$ is the pair following (i,j) in the order of (3.ii).

Remark 3.2. These three conditions are from the geometrical perspective. Every homeomorphisms on the disk would introduce the homomorphisms of fundamental group with the form of (3.i).

$\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij})$ is a loop starts at $(0,0)$ and encloses all the points and then return to $(0,0)$. Every $\rho_R(b_k)$ would leave this loop fixed and $\rho_R(b)$ would make it be a conjugation. Thus we have (3.ii).

Every G-homomorphisms under $\mathbb{Z}_p$-action would lead to (3.iii) and (3.iv).

Proof. We have

$$\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} A_{ij}x_{\mu_{ij}}A_{ij}^{-1}) = A\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij})A^{-1}$$

Thus

$$\prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} A^{-1}A_{ij}x_{\mu_{ij}}A_{ij}^{-1}A) = \prod_{i=0}^{p-1}(\prod_{j=n-1}^{0} x_{ij}) = x_{0,n-1}x_{0,n-2}\cdots x_{p-1,0}$$

We move $x_{\mu_{0,n-1}}$ from the right to the left:

$$(\star) \qquad x_{0,n-1}^{-1}(A^{-1}A_{0,n-1}x_{\mu_{0,n-1}}A_{0,n-1}^{-1}A)\cdots(A^{-1}A_{p-1,0}x_{\mu_{p-1,0}}A_{p-1,0}^{-1}A) = x_{0,n-2}\cdots x_{p-1,0}$$

The $x_{0,n-1}^{-1}$ must be absorbed. Since there is equal amount(could be negtive) of $x_{0,n-1}$ before and after each x_{ij} in each parenthesis, the $x_{0,n-1}^{-1}$ must be "transferred" alone these parentheses and eventually absorbed by a $x_{\mu_{ij}}$ that is $x_{0,n-1}$.

Suppose $x_{\mu_{0,n-1}}$ is $x_{0,n-1}$, then the transfer progress should be ended in the first parenthesis, because if $A^{-1}A_{0,n-1}$ has a multiple of $x_{0,n-1}$, then so is $A_{0,n-1}^{-1}A$, which elements in subsequent parenthesis could not absorb. Thus $x_{0,n-1}^{-1}$ would commute with $A^{-1}A_{0,n-1}$, and is absorbed by $x_{\mu_{0,n-1}}$.Therefore A^{-1} is multiple of $(\prod_{k=0}^{0+p-1} x_{k \bmod p,0})$. Now we move the $x_{0,n-2}$ from the right to the left. Eventually we would be in two cases:

(i)The process depicted above can be done till the end. Every $A^{-1}A_{ij}x_{\mu_{ij}}A_{ij}^{-1}A$ is equal to x_{ij}. Then $x_{\mu_{ij}}$ is x_{ij}, and each $A^{-1}A_{ij}$ could be reduce to identity. By (3.iii) , A and A_{ij} are both identity. Thus ρ is identity, which contradicts what we'd assumed.

(ii) If it's not case(i), we can always assume that $x_{\mu_{0,n-2}}$ is $x_{0,n-1}$ in ($\star$). Since $x_{\mu_{0,n-1}}$ is not $x_{0,n-1}$, we have two ways to reduce $x_{0,n-1}$: either $x_{\mu_{0,n-1}}A_{0,n-1}^{-1}A$ is absorbed by $A^{-1}A_{0,n-2}$, which implies (a) is true; or there is exactly one $x_{0,n-1}$ in $A^{-1}A_{0,n-1}$ which can absorb $x_{0,n-1}^{-1}$, such that $x_{0,n-1}^{-1}$ could "jump over" $x_{\mu_{0,n-1}}$, then reduce $x_{\mu_{0,n-2}}$. This means (b) is true. □

Lemma 3.3. *Under the conditions and notations of Lemma 2, if ρ satisfies Lemma 2(a),*

$$l(\rho\circ\rho_R(b_{j-1})) < l(\rho) \text{ or } l(\rho\circ\rho_R(b^{-1})) < l(\rho),$$

and if ρ satisfies Lemma 2(b),

$$l(\rho\circ\rho_R(b_{j-1}^{-1})) < l(\rho) \text{ or } l(\rho\circ\rho_R(b)) < l(\rho),$$

where l is length function, i.e., $l(\rho) = \sum_{i,j}(\text{minimum letter lengths of the words } A_{ij}x_{\mu_{ij}}A_{ij}^{-1})$.

Proof. Case(a-i): There is a j such that $x_{\mu_{ij}}A_{ij}^{-1}$ could be absorbed by $A_{i,j-1}$, which means $A_{i,j-1} = A_{ij}x_{\mu_{ij}}^{-1}B_{i,j-1}$ for any $0 \le i \le p-1$ by Lemma 2 Condition(3). We have

$$\begin{aligned}
x_{ij} &\overset{\rho}{\mapsto} A_{ij}x_{\mu_{ij}}A_{ij}^{-1}\\
x_{i,j-1} &\overset{\rho}{\mapsto} (A_{ij}x_{\mu_{ij}}^{-1}B_{i,j-1})x_{\mu_{i,j-1}}(A_{ij}x_{\mu_{ij}}^{-1}B_{i,j-1})^{-1}\\
x_{ij} &\overset{\rho_R(b_{j-1})}{\mapsto} x_{ij}x_{i,j-1}x_{ij}^{-1}\\
&\overset{\rho}{\mapsto} (A_{ij}x_{\mu_{ij}}A_{ij}^{-1})(A_{ij}x_{\mu_{ij}}^{-1}B_{i,j-1})x_{\mu_{i,j-1}}(A_{ij}x_{\mu_{ij}}^{-1}B_{i,j-1})^{-1}(A_{ij}x_{\mu_{ij}}A_{ij}^{-1})^{-1}\\
&= A_{ij}B_{i,j-1}x_{\mu_{i,j-1}}B_{i,j-1}^{-1}A_{ij}^{-1}\\
x_{i,j-1} &\overset{\rho_R(b_{j-1})}{\mapsto} x_{ij}\\
&\overset{\rho}{\mapsto} A_{ij}x_{\mu_{ij}}A_{ij}^{-1}
\end{aligned}$$

for any $0 \le i \le p-1$. Thus by comparison we get $l(\rho\circ\rho_R(b_{j-1})) < l(\rho)$.

Case(a-ii): $x_{\mu_{i0}}A_{i0}^{-1}$ could be absorbed by $A_{i+1,n-1}$, which means $A_{i+1,n-1}=A_{i0}x_{\mu_{i0}}^{-1}B_{i+1,n-1}$ for any $0\le i\le p-2$ by Lemma 2 Condition(3). We have

$$\begin{aligned}
x_{i0} &\overset{\rho}{\mapsto} A_{i0}x_{\mu_{i0}}A_{i0}^{-1}\\
x_{i+1,n-1} &\overset{\rho}{\mapsto} (A_{i0}x_{\mu_{i0}}^{-1}B_{i+1,n-1})x_{\mu_{i+1,n-1}}(A_{i0}x_{\mu_{i0}}^{-1}B_{i+1,n-1})^{-1}\\
x_{i0} &\overset{\rho_R(b^{-1})}{\mapsto} x_{i+1,0}\\
&\overset{\rho}{\mapsto} A_{i+1,0}x_{\mu_{i+1,0}}A_{i+1,0}^{-1}\\
x_{i+1,n-1} &\overset{\rho_R(b^{-1})}{\mapsto} x_{i0}x_{i+1,n-1}x_{i0}^{-1}\\
&\overset{\rho}{\mapsto} (A_{i0}x_{\mu_{i0}}A_{i0}^{-1})(A_{i0}x_{\mu_{i0}}^{-1}B_{i+1,n-1})x_{\mu_{i+1,n-1}}(A_{i0}x_{\mu_{i0}}^{-1}B_{i+1,n-1})^{-1}(A_{i0}x_{\mu_{i0}}A_{i0}^{-1})^{-1}\\
&= A_{i0}B_{i+1,n-1}x_{\mu_{i+1,n-1}}B_{i+1,n-1}^{-1}A_{i0}^{-1}
\end{aligned}$$

Since (3.iii) , letter length of $A_{i0}x_{\mu_{i0}}A_{i0}^{-1}$ is equal to $A_{i+1,0}x_{\mu_{i+1,0}}A_{i+1,0}^{-1}$. Thus we have $l(\rho\circ\rho_R(b^{-1}))<l(\rho)$.

Case(b) is similar to the discussion above. □

Theorem 3.4. *$\rho\in AutR_{pn}$ subjects to (3.i), (3.ii), (3.iii) and (3.iv) if and only if $\exists\ \sigma\in B_n^{orb}(\mathbb{C},\mathbb{Z}_p)$ such that ρ is an element of the equivalent class $\rho_R(\sigma)$.*

Proof. "If": We have discussed in Remark 3.2 .

"Only if": If $l(\rho)=pn$, then $\rho=id_{R_{pn}}\in\rho_R(e)$. Otherwise by Lemma 3.1 and Lemma 3.3 we have a $\sigma\in B_n^{orb}(\mathbb{C},\mathbb{Z}_p)$ such that $l(\rho\circ\rho_R(\sigma))<l(\rho)$. By induction we can render ρ as a product of elements from $Im(\rho_R)$. Thus the conclusion follows. □

Proof of Theorem 2.2 : Due to the discussion of Remark 3.2 , we have a map $M_n^{orb}\to\rho_R(B_n^{orb}(\mathbb{C},\mathbb{Z}_p))$. Obviously it is a well defined homomorphism. It is a monomorphism exactly as the proof of Theorem 2.6 .

On the other hand, we can realize $\rho_R(b)$ by a $\mathbb{Z}_p$-homeomorphism which fixes outside $B_{\frac{5}{3}}(0,0)$ and is $\frac{2\pi}{p}$ rotation within $B_{\frac{4}{3}}(0,0)$. And we realize $\rho_F(b_k)$ by a $\mathbb{Z}_p$-homeomorphism which interchanges q_{ik} and $q_{i,k+1}$ and fixes outside those little disks which include them. Therefore $\rho_R(B_n^{orb}(\mathbb{C},\mathbb{Z}_p))\subset M_n^{orb}$.

4. Orbit Link Group Via the Orbit Braid Representation

Just like the case of ordinary link, we could define orbit link in S^3. To find a proper definition of orbit links, we consider situations of closure of orbit braids.

The first step is to extend the G-action to S^3. There are various ways to do this, and we construct a canonical type of extension, which is about how $S^3=\partial D^4$ is attached to S^2, when we considere the CW-structure of $\mathbb{C}P^2$. If we have a boundary-keep homeomorphism g of D^2, which is a 2-cell of S^2, then we map S^3 to it by the attaching map, and then map it to itself by g, and then pull back to S^3 and get

$$(z_1,z_2)\mapsto\left(\frac{z_2}{|z_2|}g(\frac{|z_2|z_1}{z_2}),\frac{\sqrt{1-|g(\frac{|z_2|z_1}{z_2})|^2}}{|z_2|}z_2\right). \tag{4.v}$$

For example, by the following extension of $\mathbb{Z}_p$-action we regard S^3 in $\{(z_1,z_2)\in\mathbb{C}^2;|z_1|^2+|z_2|^2=1\}$. The $\mathbb{Z}_p$-action over S^3 is

$$\begin{aligned}
&\mathbb{Z}_p\times S^3\to S^3,\\
&(n,z_1,z_2)\mapsto(z_1\mathrm{e}^{\frac{2\pi ni}{p}},z_2).
\end{aligned}$$

Let me explain why it is meaningful. To make it easier to imagine, We had better treat projective space as quotient space of sphere. Then $\mathbb{C}P^2 = \{[z_1, z_2, z_3]; z_1, z_2, z_3 \in \mathbb{C}, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$. The 4-cell D^4 is homeomorphic to $\{(z_1, z_2, c); z_1, z_2 \in \mathbb{C}, c \in \mathbb{R}^+, |z_1|^2 + |z_2|^2 + c^2 = 1\}$. Then

$$\partial D^4 = S^3 = \{(z_1, z_2); z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\}$$

is directly attached to

$$\mathbb{C}P^1 = \{[z_1, z_2]; z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\} \approx S^2.$$

The 2-cell D^2 is homeomorphic to

$$(S^2)_+ = \{(z_1, b); z_1 \in \mathbb{C}, b \in \mathbb{R}^+, |z_1|^2 + b^2 = 1\}.$$

Hopf map refers to the first arrow of the following cofiber sequence:

$$\begin{aligned} S^3 &\to \mathbb{C}P^1 = S^2 \to S^2 \cup D^4 = \mathbb{C}P^2 \\ (z_1, z_2) &\mapsto [z_1, z_2] \end{aligned}$$

If we just have a homeomorphism g of D^2, which we have focused on, then we may regard D^2 as 2-cell of $\mathbb{C}P^1$ and require g to keep boundary. From above discussion we have

$$\begin{array}{ccccccccc} S^3 & \longrightarrow & (S^2)_+ & \xrightarrow{\approx} & D^2 & \xrightarrow{g} & D^2 & \xrightarrow{\approx} & (S^2)_+ \\ (z_1, z_2) & \overset{\times \frac{|z_2|}{z_2}}{\mapsto} & \left(\frac{|z_2|z_1}{z_2}, |z_2|\right) & \mapsto & \frac{|z_2|z_1}{z_2} & \mapsto & g(\frac{|z_2|z_1}{z_2}) & \mapsto & \left(g(\frac{|z_2|z_1}{z_2}), \sqrt{1 - |g(\frac{|z_2|z_1}{z_2})|^2}\right). \end{array}$$

To pull back to S^3, we multiple the twist factor $\frac{z_2}{|z_2|}$ thus we have Equation 4.v.

On the other hand, we could adopt an alternative perspective to see it more intuitively. S^1 can be given a structure of H-space: $(S^1, 1)$ with a product

$$\begin{aligned} \mu : S^1 \times S^1 &\to S^1 \\ (z_1, z_2) &\mapsto z_1 z_2, \end{aligned}$$

where 1 is a unit. The Hopf construction $H(\mu) : S^1 * S^1 \to \Sigma S^1$ is induced by the following communative diagram

$$\text{(4.vi)} \qquad \begin{array}{ccccccc} ((z_1, t), z_2) & & CS^1 \times S^1 & \longleftarrow & S^1 \times S^1 & \longrightarrow & S^1 \times CS^1 & & (z_1, (z_2, t)) \\ \downarrow & & \downarrow & & \downarrow \mu & & \downarrow & & \downarrow \\ (\mu(z_1, z_2), t) & & CS^1 & \longleftarrow & S^1 & \longrightarrow & CS^1 & & (\mu(z_1, z_2), t) \end{array}$$

where CS^1 is the cone over S^1, and $S^1 * S^1$ and ΣS^1 are the colimits of the first and the second row, respectively. $S^1 \times S^1$ is a torus, and to attach $CS^1 \times S^1$, we just attach along each meridian a disk inside the torus, and thus get a solid torus; To attach $S^1 \times CS^1$, we attach disks outside the torus along every longitude. Thus $S^1 * S^1$ is S^3. We identify S^3 with $\mathbb{R}^2 \cup \{\infty\}$ through stereographic projection and choose $z_2 = 0$ as axis, $(1, 0)$ as infinite point, and suppose the solid torus is $\{|z_1| \leq \frac{1}{2}\}$ in the S^3. Each disk inside the solid torus has the same argument of z_2.

Now assume CS^1 in the bottom left corner and the bottom right corner of (4.vi) correspond to the upper and the lower hemisphere of $\Sigma S^1 = S^2$, respectively. Then every disk inside the solid torus corresponds to the upper hemisphere by $H(\mu)$. Moreover, for each θ, $\{(z_1, z_2) \in S^3, \frac{z_2}{|z_2|} = e^{i\theta}\}$ maps to the 2-cell of S^2 and the axis $\{z_2 = 0\}$ maps to the south pole of S^2 by $H(\mu)$. Given that μ adds the arguments of two circles, These correspondences would twist as z_2 rotates. If we have a homeomorphism on the 2-cell of S^2, now we can extend it naturally, that is, it acts within every $\{(z_1, z_2) \in S^3, \frac{z_2}{|z_2|} = e^{i\theta}\}$ according to those correspondences. This approach also turns out to be (4.v).

Now we can close any geometric orbit braid of $B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)$ by identifying the initial points and the end points, and fit it in the solid torus. The closure of an orbit braid $\beta \in B_n^{orb}(\mathbb{C}, \mathbb{Z}_p)$ is denoted by $\bar{\beta}$ (by abuse of notation, β is regarded as one of its geometrical description $\widetilde{c(\beta)}$, where $\widetilde{c(\beta)} = \{\widetilde{c(\beta_1)}, \ldots, \widetilde{c(\beta_n)}\}$, $\widetilde{c(\beta_i)} = \{hc(\beta_i) | h \in G\}$, and $c(\beta_i) = \{(\beta_i(s), s) | s \in I\}$).

Observating that if b is a generator of $B_2^{orb}(\mathbb{C}, \mathbb{Z}_2)$, $\bar{b}$ has only one string, whose orbit is its own, we cannot distinguish a representative in one orbit, as what we have done in the case of orbit braids.

Definition 4.1. Given a finite group G and a topological manifold M with a G-action. L is an **orbit link** in M under the G-action if $L = (K_1, \ldots, K_{\mu(L)})$ and for each $1 \leq i \leq \mu(L)$, K_i is an continuous map from S^1 to M such that
1) For any $g \in G$ and $1 \leq i \leq \mu(L)$, there is exactly one j, $1 \leq j \leq \mu(L)$, such that $g(K_i(S^1)) = K_j(S^1)$.
2) If $\exists\ a, b \in S^1 = L^{-1}(M)$, $a \neq b$, and $1 \leq i, j \leq \mu(L)$ (i may equal j), $K_i(a) = K_j(b)$, then there is a $g \in G$ such that for any open set $U \in S^1 = K_i^{-1}(M)$ containing a, there is an open set $V \in S^1 = K_j^{-1}(M)$ containing b, such that $g|_{K_i(U)} : K_i(U) \to K_j(V)$ is a homeomorphism, and $g(K_i(a)) = K_j(b)$.
Let $(K_i, K_j) = \{g \in G | g(K_i) = K_j\}$. If $(K_i, K_j) \neq \emptyset$, we say K_i and K_j belong to the same orbit.

Obviously, (K_i, K_i) is a subgroup of G. Moreover, under the precedent G-action, (K_i, K_i) is a cyclic group. This property would be proved after the proof of Axander theorem for orbit links.

In the following part we would focus on a particular type of orbit links, which are in S^3 under a G-action extended by a G-action on 2-dimension disk. The twist mentioned above doesn't really matter, so we work with a simplified form of Equation 4.v:

Proposition 4.2. *Suppose homeomorphism $g(z)$ of 2-dimension disk $\{|z|^2 = x^2 + y^2 \leq 1\}$ to itself satisfies $g(\partial D^2) = \partial D^2$ and $g|_{\partial D^2}$ is a homeomorphism. Then*

$$E(g) : S^3 \to S^3$$
$$(z_1, z_2) \mapsto (g(z_1), \frac{\sqrt{1 - |g(z_1)|^2}}{|z_2|} z_2)$$

is a homeomorphism of S^3.

Proof. We regard S^3 in the following way:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 = 1\},$$

and set

$$\begin{cases} z_1 = x_1 + iy_1 = r_1 \mathrm{e}^{\mathrm{i}\theta_1} \\ z_2 = x_2 + iy_2 = r_2 \mathrm{e}^{\mathrm{i}\theta_2} \end{cases}$$

Than $(r_1, \theta_1, \theta_2)$ is a smooth chart of S^3 around any $p \in \{(z_1, z_2) \in \mathbb{C}^2; z_2 \neq 0\}$, and (r_1, θ_1) is a smooth chart of D^2 around any $p \in \{|z_1|^2 \leq 1; z_1 \neq 0\}$. Around $p \in \{(z_1, z_2) \in \mathbb{C}^2; z_1 \neq 0, z_2 \neq 0\}$, $p = (r_1, \theta_1, r_2, \theta_2)$, $\tilde{g}(r_1, \theta_1) = g(z_1)$ is a homeomorphism thus $E(g)(z_1, z_2) = \widetilde{E(g)}(r_1, \theta_1, \theta_2) = (\tilde{g}(r_1, \theta_1), \theta_2)$ is a homeomorphism.

For p whose $z_1 = 0$, we choose a neighborhood that is homeomorphic to a cylinder $U \times I$. $E(g)|_{U \times I}$ is $E(g)(z_1, \theta_2) = (g(z_1), \theta_2)$.

For p whose $z_2 = 0$, we choose a neighborhood that is a ball formed by a bunch of half disks with each disk the same θ_2 and $E(h)$ keeps each disk to the disks around $E(h)(p)$. Therefore, $E(g)$ is smooth on S^3. □

Example 4.3.

Proposition 4.4. *If D^2 admits an effective G-action such that every $g \in G$ satisfies the properties in Proposition 4.2, then for any $\beta \in \mathcal{B}_n^{orb}(D^2, G)$, $\bar{\beta}$ is an orbit link in S^3 that admits $E(G)$-action.*

Proof. Assum that $\beta(0) = \mathbf{x} = (x_1, \ldots, x_n)$, $\beta(1) = g\mathbf{x}_\sigma = (g_1 x_{\sigma(1)}, \ldots, g_n x_{\sigma(n)})$, and $\widetilde{c(\beta)}$ is a representative. Let $\overline{\beta_i} : I \to S^3$ be the "closure" of the i-th component of β, $o(\sigma)$ be the order of σ in

Σ, and $l_i = o(g_i g_{\sigma(i)} g_{\sigma^2(i)} \cdots g_{\sigma^{o(\sigma)-1}(i)})$ be the order of $g_i g_{\sigma(i)} g_{\sigma^2(i)} \cdots g_{\sigma^{o(\sigma)-1}(i)}$ in G. Define

$$K_i : S^1 \to S^3$$

$$z = \mathrm{e}^{\mathrm{i}\theta} \mapsto \begin{cases} \overline{\beta_i}(\frac{l_i\theta}{2\pi}), & 0 \leq \frac{\theta}{2\pi} \leq \frac{1}{l_i} \\ g_i\overline{\beta_{\sigma(i)}}(\frac{l_i\theta}{2\pi} - 1), & \frac{1}{l_i} \leq \frac{\theta}{2\pi} \leq \frac{2}{l_i} \\ \cdots & \\ g_i g_{\sigma(i)} g_{\sigma^2(i)} \cdots g_{\sigma^{(l_i-2)}(i)} \overline{\beta_{\sigma^{(l_i-1)}(i)}}(\frac{l_i\theta}{2\pi} - (l_i - 1)), & \frac{l_i-1}{l_i} \leq \frac{\theta}{2\pi} \leq 1 \end{cases}$$

Now $\{hK_i | h \in E(G), 1 \leq i \leq n\}$ includes all the points of $\bar{\beta}$. Remove overlaps, Then we get the form $\bar{\beta} = (K_1, \ldots, K_{\mu(\bar{\beta})})$.

The first property of Definition 4.1 is easy to check. If $\exists\ a, b \in S^1$, $a \neq b$, and $1 \leq i, j \leq \mu(L)$, $K_i(a) = K_j(b)$, then there must be a fix point $\beta_k(s)$ in D^2 for some $g \in G$, which corresponds to $K_i(a) = K_j(b)$ in S^3 after closing the braid. Suppose $g\beta_k$ and $h\beta_k$ correspond to (part of) K_i and K_j which contain $K_i(a)$ and $K_j(b)$, then $E(hg^{-1}) \in (K_i, K_j)$ meets the second requirement of Definition 4.1. □

Now we intend to present every smooth orbit link, which is in S^3 under a G-action extended by a G-action on 2-dimension disk, as the closure of an orbit braid. The equivalence of orbit links is induced by isotopy:

Definition 4.5. Two (piecewise) smooth orbit links L and L' in M under the G-action are *equivalent* if there is a homotopy map $T = (T_1, \ldots, T_{\mu(L)})$, such that for each i, $T_i : I \times S^1 \to M$ is (piecewise with respect to S^1) smooth, and $T_i(0, s) = K_i(s)$, $T_i(1, s) = K_i'(s)$, and for each $t \in I$, $T(t, \cdot)$ is an orbit link in M under the G-action, which satifies $(T_i(t, \cdot), T_j(t, \cdot)) = (K_i, K_j)$.

In the ordinary case, smooth links could be identified with polygonal links. In orbit case, however, this simplification generally cannot be made, since we cannot expect orbits of a geodesic to be geodesics. Thus we use piecewise smooth curves to substitute polygonal curves. By polishing finite many singular points, the equivalence classes of piecewise smooth orbit links can be mapped to the equivalence classes of smooth orbit links. This map is surjective and injective, since smooth links are piecewise smooth, and smooth map T_i is piecewise smooth.

We try to define a "generator" to describe the element of deformation when applying isotopy to a piecewise smooth orbit link:

Definition 4.6. If $a_0, a_2, \cdots$ are points in a smooth manifold M which admits a G-action, then $[a_0, \cdots, a_n]$ denotes a smooth n-simplex whose vertices are points $a_0, \cdots, a_n$. $[a_1, \cdots, a_n]L$ denotes $[a_1, \cdots, a_n] \cap L$.

Let L be a piecewise smooth orbit link with a smooth curve $[a, c]$, and b is a point outside L. Suppose there is a $[a, b, c]$ such that

$$[a, b, c]L = [a, c].$$

Then we say that operation $\mathscr{E}^b_{ac}$ *is applicable to* L, if

$$\mathscr{E}^b_{ac}L = L - [a, c] + [a, b] + [b, c]$$

is equivalent to L in terms of Definition 4.5. Such operations and their inverse are called *elementary deformations* of orbit links.

Theorem 4.7. *Every smooth orbit link in S^3 under an $E(G)$-action is equivalent to a closed orbit braid in D^2 under the G-action.*

Proof. In ordinary case the *axis* can be chosen arbitrarily, only required not to meet the link L. In orbit case, however, the axis $\{z_2 = 0\}$ is fixed, hence an aditional step is needed to move L away from the axis. If L has one component $K_1(z) \subset \{z_2 = 0\}$, namely it is equivalent to $K_1(z) = (z, 0)$, then $T_1(t, z) = ((1 - \frac{\epsilon t}{2})z, \sqrt{\epsilon t - \frac{(\epsilon t)^2}{2}})$ fulfil the goal if ϵ is sufficiently small.

Now assume it is not this case. Suppose

$$\begin{aligned}\widetilde{K_i}:&\mathbb{R}\to S^3\\ &s\mapsto K_i(\mathrm{e}^{\mathrm{i}s}),\end{aligned}$$

has $\widetilde{K_i}(s_0)\in\{z_2=0\}$. Choose x smaller than but very close to $\sup\{s\in\mathbb{R}; s<s_0, \widetilde{K_i}(s)\notin\{z_2=0\}\}$, and choose y larger than but very close to $\inf\{s\in\mathbb{R}; s>s_0, \widetilde{K_i}(s)\notin\{z_2=0\}\}$, such that $a=\widetilde{K_i}(x), c=\widetilde{K_i}(y)\notin\{z_2=0\}$. Then choose a point b not in $L\cup\{z_2=0\}$ but sufficiently close to $[a,c]$, such that $\mathscr{E}^b_{ac}$ is applicable to L, and $(\partial[a,b,c]-[a,c])\cap\{z_2=0\}=\emptyset$. Then we also have $E(G)(\partial[a,b,c]-[a,c])\cap\{z_2=0\}=\emptyset$, since $E(G)(\{z_2=0\})\subset\{z_2=0\}$.

Since the set of all open sets of $\{z_2=0\}\}$ is countable, we could find a cover of $L\cap\{z_2=0\}$, composed of countable smooth curves $[a_i,c_i]$ with vertices a_i,c_i not in $\{z_2=0\}$. Find b_i for each curve step by step such that $\mathscr{E}^{b_i}_{a_ic_i}$ is applicable to $\mathscr{E}^{b_{i-1}}_{a_{i-1}c_{i-1}}\cdots\mathscr{E}^{b_1}_{a_1c_1}L$, and let homotopy $T^i(t)$ denotes this equivalence. Then

$$T(t)=\begin{cases}T^1(2t), & 0\le t\le\frac{1}{2}\\ \cdots & \\ T(t)=T^i(2^it+2-2^i), & 1-2^{1-i}\le t\le 1-2^{-i}\\ \cdots & \end{cases}\tag{4.vii}$$

alters L to an equivalent link without touching the axis.

Now we can regard this link as in $\mathbb{R}^3$, where the axis $\{z_2=0\}$ corresponds to the z-axis, and $\{z_1=0\}$ corresponds to a circle, and $\{(z_1,z_2)\in S^3, z_2/|z_2|=\mathrm{e}^{\mathrm{i}\theta_0}\}$ corresponds to $\{\theta=\theta_0\}$ in cylindrical coordinates. For this orbit link, we expect that none of its smooth pieces has its tangent $\dot{K}_i(s)$ coplanar with the axis, except countable many end points. We apply a similar approach to achieve this goal:

Suppose with respect to the z-axis, the *position* of the curve at s_0

$$\left\langle\dot{\widetilde{K_i}}(s_0), \boldsymbol{e_\theta}(\widetilde{K_i}(s_0))\right\rangle$$

is zero, where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product, and $\boldsymbol{e_\theta}(x,y,z)=\boldsymbol{(y,-x,0)}$ is a vector. Choose x smaller than but very close to

$$\sup\{s\in\mathbb{R}; s<s_0, \left\langle\dot{\widetilde{K_i}}(s), \boldsymbol{e_\theta}(\widetilde{K_i}(s))\right\rangle\neq 0\},$$

and choose y larger than but very close to

$$\inf\{s\in\mathbb{R}; s>s_0, \left\langle\dot{\widetilde{K_i}}(s), \boldsymbol{e_\theta}(\widetilde{K_i}(s))\right\rangle\neq 0\},$$

such that $a=\widetilde{K_i}(x), c=\widetilde{K_i}(y)$ have their tangent not coplanar with the axis. Then choose a point b sufficiently close to $[a,c]$, such that $\mathscr{E}^b_{ac}$ is applicable to L, and $(\partial[a,b,c]-[a,c])$ have their tangent not coplanar with the axis, except b. Then we also have tangents of $E(G)(\partial[a,b,c]-[a,c]-b)$ not coplanar with the axis, since for each θ, $E(G)$ maps $\{(z_1,z_2)\in S^3, z_2/|z_2|=\mathrm{e}^{\mathrm{i}\theta}\}$ to itself, which corresponds to a plane through z-axis in $\mathbb{R}^3$.

Then, since the orbit link is compact, exactly the same approach of Equation 4.vii can be applied to acquire an equivalent orbit link which is composed of countable many smooth pieces, each with a constant position.

After then, we apply Lemma 2.1.2 of [BC74] with some simplification and necessary modifications, to deform the link such that any smooth piece of it has a positive position. For each point of a negative curve $[a,c]$, draw a smooth curve inside the plane through this point and the z-axis, joining this point to the z-axis, and avoiding any other part of the link. This is possible, since the orbit link is compact, and its intersection with the plane through the z-axis is finite many points.

Suppose a is joined to $b\in\{x=0,y=0\}$ by $[a,b]$. Thicken this 1-simplex and get a 2-simplex $[a,b,a_1]$ where $a_1\in[a,c]$, such that $[a,b,a_1]\cap L=[a,a_1]$ and $[a_1,b]\subset\{\theta=\theta(a_1)\}$. We propose an isotopy such that $\mathscr{E}^b_{aa_i}$ is applicable to L. □

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