CONTACT GEOMETRY FROM AN UNDERGRADUATE'S VIEWPOINTS

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1. Introduction

I think details of a definition are important, since they may show some implicit ideas of the new structures. This note is mainly based on [Etn03] [Etn01] and [Hon04].

A brief index of this note (in progress):

Cooriented contact structure means a global contact form (Proposition 2.1), and why the 3-manifold should be assumed orientable, other than the contact structure (Remark 2.2) ;

The definition of the framing (Definition 3.1), twisting number (Definition 3.2), Thurston-Bennequin invariant (Definition 3.3), rotation number (Definition 3.4), and why they are well defined (Proposition 3.5 and Proposition 3.7);

Represent the Thurston-Bennequin invariant as the linking number (Proposition 3.6);

Another definition of rotation number induced by the Seifert surface (Remark 3.8), and why the contact structure over the Seifert surface is trivial, from a point of homotopy theory (3.10).

2. contact structure

Contact structures ξ on 3-manifolds M are usually assumed to be *cooriented*, i.e., M and ξ is oriented and $\alpha \wedge \mathrm{d}\alpha > 0$. In such case α can be chosen as a global 1-form:

Proposition 2.1. *If ξ is a smooth 2-plane field with an orientation induced by TM's, then we can find a 1-form α defined on all of M such that ξ is the kernal of α (HW 1 of* [Hon04] *and Exercise 2.3 and 2.4 of* [Etn01] *).*

Proof. For an altas $\{(\varphi_\lambda, U_\lambda) | \lambda \in \Lambda\}$ of M, there is a local trivialization:

$$\begin{array}{ccccc} TM|_{U_\lambda} & \xrightarrow{\phi_\lambda} & U_\lambda \times \mathbb{R}^3 & \longrightarrow & \varphi_\lambda(U_\lambda) \times \mathbb{R}^3 \\ \uparrow & & \uparrow i & & \uparrow i \\ \xi|_{U_\lambda} & \xrightarrow{\phi_\lambda|_\xi} & U_\lambda \times \mathbb{R}^2 & \longrightarrow & \varphi_\lambda(U_\lambda) \times \mathbb{R}^2 \end{array}$$

such that $g_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1}$ preserves the orientation of M.

Since ξ is smooth, there is a smooth non-zero vector field defined on $\varphi_\lambda(U_\lambda)$ tangent to $(\varphi_\lambda)_*\xi \subset \mathbb{R}^3$, such that it and a positively oriented basis of $(\varphi_\lambda)_*\xi \subset \mathbb{R}^3$ form a positively oriented basis of $(\varphi_\lambda)_*TM$. Regard it as a 1-form in $T^*\varphi_\lambda(U_\lambda)$ and pull it back to U_λ then we have desired α_λ locally.

There is a partition of unity $\{\rho_\lambda | \lambda \in \Lambda\}$ and $\sum \rho_\lambda \alpha_\lambda$ gives the desired global 1-form, since directions of $\{\alpha_\lambda | \lambda \in \Lambda\}$ correspond on the intersections of $\{U_\lambda | \lambda \in \Lambda\}$ thus their sum would never be zero. □

Remark 2.2. TM should been assumed to be orientable, since the orientation of ξ cannot decide an orientation of TM, without which we cannot choose local 1-forms uniformly.

This is a counter-example in Figure 1: a Klein bottle $(\mathbb{R}^2 \times I)/\sim$ where $(x, y, 0) \sim (-x, y, 1)$, with

$$\xi = \mathbb{R}\{\frac{\partial}{\partial t}, \sin(2\pi t)\frac{\partial}{\partial x} + \cos(2\pi t)\frac{\partial}{\partial y}\}. \tag{2.i}$$

Check it is a contact structure: a 1-form

$$\alpha = \cos(2\pi t)\mathrm{d}x - \sin(2\pi t)\mathrm{d}y \tag{2.ii}$$

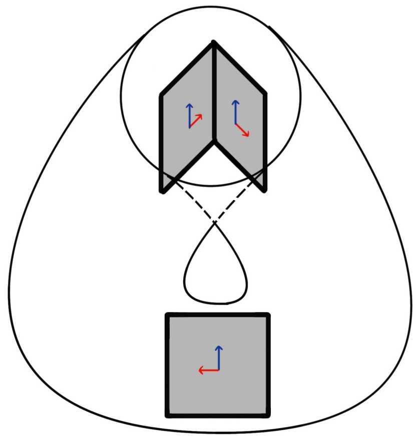


FIGURE 1. The red arrow refers to $\frac{\partial}{\partial t}$.

defined on $\mathbb{R}^2 \times (0,1) \subset (\mathbb{R}^2 \times I)/\sim$ satisfies

$$\begin{aligned}\alpha \wedge \mathrm{d}\alpha &= (\cos(2\pi t)\mathrm{d}x - \sin(2\pi t)\mathrm{d}y) \wedge (-\sin(2\pi t)\mathrm{d}t \wedge \mathrm{d}x - \cos(2\pi t)\frac{\partial}{\partial y}\mathrm{d}t \wedge \mathrm{d}y) \\ &= \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t.\end{aligned}$$

Slightly move α and then we get another local 1-form covering $\mathbb{R}^2 \times \{0\} \subset (\mathbb{R}^2 \times I)/\sim$. However these local forms cannot be extended into a global form.

3. CLASSICAL INVARIENTS OF LEGENDRIAN KNOTS

Definition 3.1. An isomorphism between the normal bundle ν of a knot L and the trivial bundle of L is called a *framing* of L.

A 2-framing of L is equivalent to:

- a section s of the normal bundle ν, since it corresponds to a fixed vector in the fiber $\mathbb{R}^2$ of the trivial bundle;
- a homeomorphism between the tubular neighborhood N of L and a solid torus, in which the closed curve corresponding to the section s, locating on ∂N, is identified with a latitude of T^2.

Use the second explanation:

Definition 3.2. Suppose $\mathcal{F}$ and $\mathcal{F}'$ are two framings of L, and ∂N is identified with T^2 by $\mathcal{F}$. Assume that the first component of $H_1(\partial N) = \mathbb{Z} \bigoplus \mathbb{Z}$ is longitude and the second is latitude. If the closed curve corresponding to $\mathcal{F}'$, locating on ∂N, is a representative of $(n, 1) \in H_1(\partial N)$, then the *twisting number* $tw(L, \mathcal{F}, \mathcal{F}') = n$.

Use the first explanation:

Definition 3.3. Given an orientation of L, every orientable 2-dimension subbundle of the normal bundle ν decides a 2-framing of L. Since the contact structure ξ is orientable, it induces a canonical framing $\mathcal{F}_\xi$ which is called *Thurston-Bennequin framing* $\mathcal{F}_\xi$. If L is the boundary of a *Seifert surface*, which is a compact orientable surface Σ, then $T\Sigma|_L$ is orientable and induces a framing $\mathcal{F}_\Sigma$. The *Thurston-Bennequin invariant* $tb(L)$ is defined as

$$tb(L) = tw(L, \mathcal{F}_\Sigma, \mathcal{F}_\xi) \tag{3.iii}$$

Since L is homeomorphic to S^1 and any orientable vector bundle of S^1 is isomorphic to the trivial bundle, $\xi|_L$ is isomorphic to the trivial bundle $L \times \mathbb{R}^2$. Under this isomorphism, the winding number of L can be defined as another invarient:

Definition 3.4. Choose a parameterization $\phi(\theta)$ of L and an isomorphism $g : \xi|_L \to L \times \mathbb{R}^2$, induced by a trivialization of $\xi|_M$, where M is a contractible space. Then the degree of

$$S^1 \longrightarrow \xi|_L \xrightarrow{g} L \times (\mathbb{R}^2 - 0) \xrightarrow{projection} (\mathbb{R}^2 - 0) \tag{3.iv}$$

$$\theta \mapsto (\phi(\theta), \phi'(\theta)) \mapsto g(\phi(\theta), \phi'(\theta)) \mapsto pr_2(g(\phi(\theta), \phi'(\theta))) \tag{3.v}$$

is called the *rotation number* $r(L)$.

Now we give some details to prove that these definitions are well-defined.

Proposition 3.5. *(HW 5 of [Hon04]) Thurston-Bennequin invariant $tb(L)$ does not depend on the choice of Seifert surface Σ.*

Proof. Since the normal framing induced from the contact structure is hard to detect, we just consider $t(L, \mathcal{F}_\Sigma)$ and $t(L, \mathcal{F}_{\Sigma'})$ induced by two different Seifert surfaces. It's sufficient to prove that the boundary of one Seifert surface couldn't wind around another's.

Thicken the knot L a little, and get a tubular neighborhood N of it, such that each of these two Seifert surfaces intersects with ∂N at a knot. Let $M = S^3 - N$. By Zig-Zag Lemma, there is an long exact sequence of relative homology group

$$H_2(M) \to H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \to H_1(M) \tag{3.vi}$$

First, M is homotopic to $S^3 - S^1 = \mathbb{R}^3 - \{x = y = 0\}$, which is homotopic to S^1. Thus $H_2(M) = 0$ and $H_1(M)$ is one dimension. Moreover, the longitude of ∂M corresponds to a non-trivial close chain in M. Thus if the first component of $H_1(\partial M)$ is longitude and the second is latitude, then

$$H_1(\partial M) \to H_1(M)$$
$$(n, m) \mapsto n.$$

Next, $[\Sigma - N]$ is an element of $H_2(M, \partial M)$, and $\Sigma - N$ is a 2-chain of M, and it's boundary is a 1-chain of ∂M. Thus $[\partial(\Sigma - N)]$ is the image of $[\Sigma - N]$ under ∂.

As a result, if ∂N is identified with a torus by $\mathcal{F}_\Sigma$, then the 1-chain $\partial(\Sigma' - N)$ must represent $(0, 1)$ in $H_1(\partial M)$. Thus the closed curves corresponding to the normal framing, locating on the ∂N identified by $\mathcal{F}_\Sigma$ and $\mathcal{F}_{\Sigma'}$ respectively, represent the same element in $H_1(T^2)$. ☐

Proposition 3.6. *(Remark 2.14 of [Etn03]) Thurston-Bennequin invariant $tb(L) = lk(L, L')$, where L' is obtained from L by pushing it slightly in the direction of $\mathcal{F}_\xi$.*

Proof. First, we consider L'' which is obtained from L by pushing it slightly in the direction of $\mathcal{F}_\Sigma$. Recall how Seifert algorithm produces a Seifert surface: resolves the crossings of the projection of the knot with respect to the orientation of the knot and yields several oriented simple closed curves; attaches a disk alone each closed curve and concatenates them at the crossings by seeing them as twists of the surface.

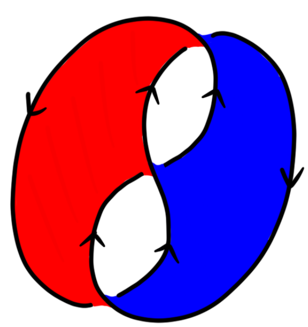


Figure 2. The Seifert surface of the eight knot

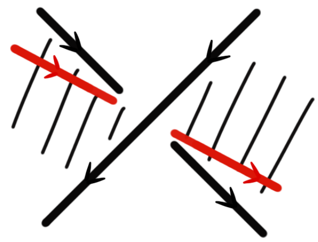


FIGURE 3. L'' (red) is between the upper string and the lower string of L (black) by the construction of Seifert surface.

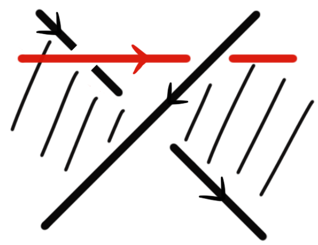


FIGURE 4

Outside a neighboorhood of the self crossings of L, it is easy to let L'' without crossings of L and L''. At a self crossing of L, one can persturb L'' such that there will be one left handed crossing and one right handed crossing of L and L'' (see Figure 3 and Figure 4).

In conclusion, $lk(L, L'') = 0$.

Now, $tb(L) = tw(L, \mathcal{F}_\Sigma, \mathcal{F}_\xi)$ actually measures how many times L' twists around L with respect to L''. Thus we can move L' isotopically such that L' overlaps L'' for most of the time, and for a sufficiently small time period $(0, \delta)$, L' twists $tb(L)$ times around L while L'' twists 0 time around L. Every time L' twists around L it would yeild two right handed crossings, thus it contributes one to $lk(L, L')$, whereas the crossings of $(\delta, 1)$ add up to zero. Therefore $tb(L) = lk(L, L')$. □

Proposition 3.7. *Rotation number $r(L)$ defined in Definition 3.4 does not depend on the choice of the trivialization of ξ (HW 6 of* [Hon04]*)*

Proof. Suppose there are two trivializations of $\xi|_M$, namely two isomorphisms of bundle $g, h : \xi|_M \to M \times \mathbb{R}^2$.

Regard $L(s)$ as the parameterization of L. Fix a non-zero point v of $\mathbb{R}^2$, the fiber of ξ. Define

$$f : S^1 \to (\mathbb{R}^2 - 0) \tag{3.vii}$$

$$s \mapsto pr_2(g \circ h^{-1}(L(s), v)), \tag{3.viii}$$

whose degree meassures how many times the trivialization h twists with respect to g, as the point of the base space goes along the knot L. Since M is contractible, there is a deformation map $H : M \times I \to M$ with $H(x, 0) = x$ and $H(x, 1) = L(0)$. Now

$$f_t : S^1 \to (\mathbb{R}^2 - 0) \tag{3.ix}$$

$$s \mapsto pr_2(g \circ h^{-1}(H(L(s), t), v)) \tag{3.x}$$

says that $\deg(f) = 0$.

By comparison with the equation Equation 3.iv in Definition 3.4, it turns out that the the rotation number is well defined. □

Remark 3.8. An alternative definition of rotation number is by the trivialization of $\xi|_\Sigma$, where Σ is the Seifert surface of L. The fact that any orientable 2 dimensional bundle over a surface with boundary is trivial, is based on the following two lemmas.

At the first glance, this definition is unclear, since $\xi|_\Sigma$ is just **isomorphic** to a trivial bundle. For different trivializations, the fiber may twist along a path of the base space. For example, we can choose an eight knot whose front projection is flat enough, such that ξ restricted on its Seifert surface is very close to $\mathrm{d}z = 0$. Let $w = \frac{\partial}{\partial y}$ and it gives a triviliazation of ξ. Then choose another trivialization given by

$$v(\theta) = \begin{cases} \frac{\partial}{\partial y}\cos(2\theta) + \frac{\partial}{\partial x}\sin(2\theta), & 0 \le \theta \le \pi \\ \frac{\partial}{\partial y}, & others \end{cases} \tag{3.xi}$$

where $0 \le \theta \le \pi$ is the upper half part of Figure 5.

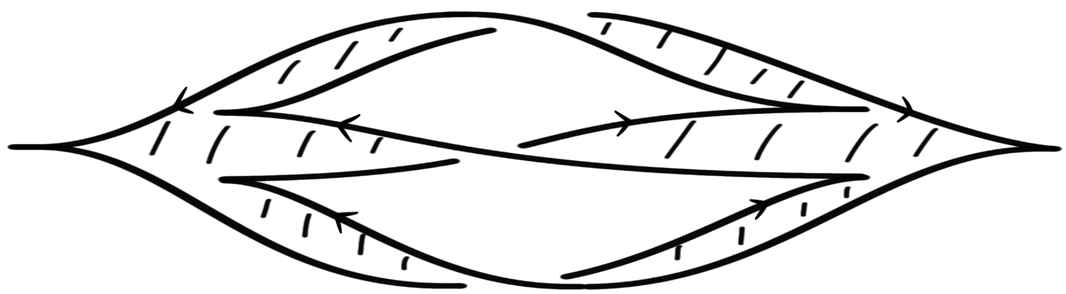


FIGURE 5

Now it is easy to see that the rotation numbers induced by these two trivializations are the same, since there are two pieces of string in the opposite direction in this upper half part.

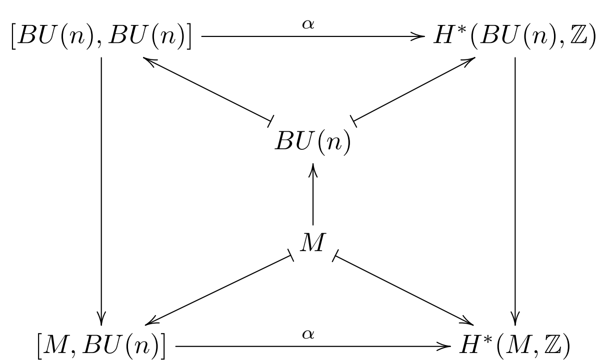
Lemma 3.9. *For a surface M with boundary, $H^2(M, \mathbb{Z}) = 0$.*

In particular, a Seifert surface is homotopic to a graph. For example, choose a vertex for each disk, and add an edge for each crossing of the knot, which is the boundary of the Seifert surface. Then we obtain a strong deformation retract of the Seifert surface of the eight knot (Figure 2).

Lemma 3.10. *A complex line bundle with vanishing first Chern class is trivial.*

Let $Vect_n$ be a contravariant functor sends a topological space to the isomorphism class of its n-dimensional complex vector bundle. $Vect_n(M)$ is in one-to-one corespondence with $[M, BU(n)]$, by the classifiction theorem. A *characteristic class* of n-dimensional complex vector bundle is a natural transformation from $[-, BU(n)]$ to $H^*(-, \mathbb{Z})$. Then we have an implicit commutative diagram:

(3.xii)



Thus this natural transformation is decided by the image of $id \in [BU(n), BU(n)]$ in $H^*(BU(n), \mathbb{Z})$. Thus characteristic classes are in one-to-one corespondence with $H^*(BU(n), \mathbb{Z})$.

When $n = 1$, from the fibration

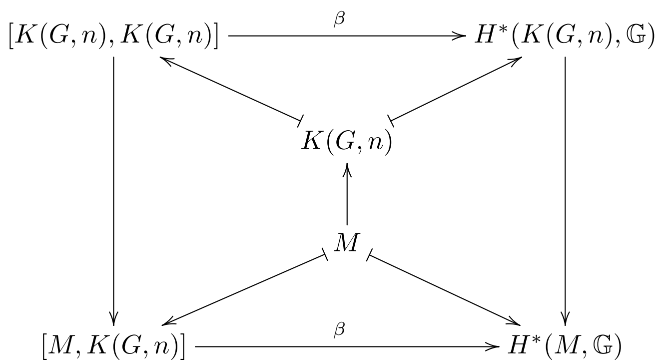
$$\begin{array}{ccc} U(1) = S^1 & \longrightarrow & EU(1) = S^\infty \\ & & \downarrow \\ & & BU(1) = \mathbb{C}P^\infty \end{array}$$

and the 2 dimensional spectral sequence we have $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[c_1]$, where c_1 has degree two and is called the *universal first Chern class*. It corresponds a natural transformation from $[-, BU(1)]$ to $H^2(-, \mathbb{Z})$.

Now, given a complex line bundle over M, the corresponding element in $[M, BU(1)]$, which is called the classifying map, has an image in $H^2(M, \mathbb{Z})$, which is also the pullback of c_1 via the classifying map. It is called the first Chern class of the complex line bundle.

Here comes an interesting thing: the essence of Lemma 3.10 is showed in a very similar commutative diagram from homotopy theory:

(3.xiii)



where $[-, K(G,n)]$ is a functor sends a CW complex M, to the homotopy class of maps from it to an Eilenberg-MacLane space, whose $(n-1)$-skeleton is a point. By the cellular approximation theorem, every such homotopy class identifies a map from n-cells of M to $\pi_n(K(G,n)) = G$, via the degree restricted on each n-cell. Under this map, the image of ∂e_{n+1} for a $(n+1)$-cell e_{n+1} of M, must add up to $1 \in G$. Thus it is a G-coefficeient close n-chain.

Conversly, each element α^n of $H^*(M, \mathbb{G})$ identifies an element of $[M, K(G,n)]$: $(n-1)$-skeleton of M is mapped to the 0-cell of $K(G,n)$; By the construction of $K(G,n)$ (the colimit of balls, whose each n-cell is a generator of G), every n-cell of M can be mapped to $K(G,n)$ via α^n (denoted by f); Since $\alpha^n(\partial e_{n+1})$ is $1 \in G$, $f(\partial e_{n+1}) \subset K(G,n)$ has $(n+1)$-cell attached to it, thus f can be extended to $(n+1)$-skeleton of M.

Therefore, β depicted in Equation 3.xiii is a natural isomorphism. In particular, by the exact sequence

$$\pi_n(S^\infty) \to \pi_n(\mathbb{C}P^\infty) \to \pi_{n-1}(S^1) \to \pi_{n-1}(S^\infty), \tag{3.xiv}$$

and the telescope construction of S^∞, we conclude that $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. And obviously, $\beta : [K(\mathbb{Z},2), K(\mathbb{Z},2)] \to H^*(K(\mathbb{Z},2), \mathbb{Z})$ maps identity to c_1, which is also called the *universal cohomology class*. Thus Equation 3.xii and Equation 3.xiii are exactly the same when the dimension is suitable.

Thus if a complex bundle over M has vanishing first Chern class, its classifying map is homotopic to a constant map. Therefore, this bundle is **isomorphic** to the trivial bundle.

References

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